A NOTE ON ALGEBRAIC RICCATI EQUATIONS ASSOCIATED WITH REDUCIBLE SINGULAR M-MATRICES

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Abstract. We prove a conjecture about the minimal nonnegative solutions of algebraic Riccati equations associated with reducible singular M-matrices. The result enhances our understanding of the behaviour of doubling algorithms for finding the minimal nonnegative solutions.

1. Introduction

For the algebraic Riccati equation

$$XCX - XD - AX + B = 0, \tag{1}$$

where A, B, C, D are real matrices of sizes $m \times m, m \times n, n \times m, n \times n$, respectively, a systematic study was done in [1] when

$$K = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \tag{2}$$

is a nonsingular M-matrix or an irreducible singular M-matrix. The study was recently extended in [2] to reducible singular M-matrices under suitable assumptions.

A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive, so any Z-matrix A can be written as $sI - B$ with $B \geq 0$. A Z-matrix A is called an M-matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ is the spectral radius; it is a singular M-matrix if $s = \rho(B)$ and a nonsingular M-matrix if $s > \rho(B)$.

Some regularity assumption is needed to guarantee the existence of a solution of the equation (1) associated with the M-matrix K. An M-matrix A is said to be regular if $Av \geq 0$ for some $v > 0$.

The following result is proved in [2].

Theorem 1. *Suppose the matrix K in (2) is a regular M-matrix. Then (1) has a minimal nonnegative solution Φ and $D - C\Phi$ is a regular M-matrix, and the dual equation*

$$YBY - YA - DY + C = 0, \tag{3}$$

has a minimal nonnegative solution Ψ and $A - B\Psi$ is a regular M-matrix. Moreover, $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are both regular M-matrices.

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Associated with the matrix K in (2) is the matrix

$$H = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} K = \begin{bmatrix} D & -C \\ B & -A \end{bmatrix}. \tag{4}$$

When the matrix K in (2) is regular singular M-matrix, the following assumption has been introduced in [2].

Assumption 1. *The matrix H in* (4) *has only one linearly independent eigenvector corresponding to the zero eigenvalue of multiplicity $r \geq 1$.*

It is known [1] that the assumption is satisfied with $r = 2$ when K is an irreducible singular M-matrix.

Under Assumption 1, there are nonnegative nonzero vectors $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, where $u_1, v_1 \in \mathbb{R}^n$ and $u_2, v_2 \in \mathbb{R}^m$, such that

$$K \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0, \quad [u_1^T \; u_2^T] K = 0.$$

They are each unique up to a scalar multiple [2].

The purpose of this note is to provide an affirmative answer to a conjecture in [2], regarding the matrices $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$.

2. The result

The following result was conjectured to be true in [2], and was proved under the restrictive assumption that at least one of Φ and Ψ is positive.

Theorem 2. *Let K be a regular singular M-matrix with Assumption 1. If $u_1^T v_1 \neq u_2^T v_2$, then $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are nonsingular M-matrices.*

Proof. By Theorem 1, $I_m - \Phi\Psi$ and $I_n - \Psi\Phi$ are both M-matrices. So we just need to show they are nonsingular when $u_1^T v_1 \neq u_2^T v_2$. Since $I_n - \Psi\Phi$ is nonsingular if and only if $I_m - \Phi\Psi$ is nonsingular, we only need to show $I_m - \Phi\Psi$ is nonsingular. In view of

$$\begin{bmatrix} I_n & 0 \\ -\Phi & I_m \end{bmatrix} \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} = \begin{bmatrix} I_n & \Psi \\ 0 & I_m - \Phi\Psi \end{bmatrix},$$

we need to show that the matrix

$$\begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix}$$

is nonsingular. Since Φ and Ψ are solutions of (1) and (3), respectively, it is easily verified that [2]

$$\begin{bmatrix} D & -C \\ B & -A \end{bmatrix} \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} = \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -S \end{bmatrix}, \tag{5}$$

where $R = D - C\Phi$ and $S = A - B\Psi$ are M-matrices. Therefore, the eigenvalues of R and S are all in the closed right half plane, with 0 being the only possible eigenvalue on the imaginary axis. When $u_1^T v_1 \neq u_2^T v_2$, we know from [2] that one of the matrices R and S is singular and the other is nonsingular. It follows that the matrices R and $-S$ have no eigenvalues in common.

Let

$$W = \operatorname{Ker} \begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix}.$$

We need to show $W = \{0\}$.

For any $x \in W$, post-multiplying (5) by x shows that $Tx \in W$, where T is the linear transformation from $\mathbb{C}^{m+n}$ to $\mathbb{C}^{m+n}$, defined by

$$T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where $y_1 \in \mathbb{C}^n$ and $y_2 \in \mathbb{C}^m$. Thus W is an invariant subspace of the linear transformation T. Suppose $W \neq \{0\}$. Then we have $0 \neq w \in W$ such that $Tw = \lambda w$ for some $\lambda \in \mathbb{C}$. Write $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, where $w_1 \in \mathbb{C}^n$ and $w_2 \in \mathbb{C}^m$. We then have $Rw_1 = \lambda w_1$ and $(-S)w_2 = \lambda w_2$. Since R and $-S$ have no eigenvalues in common, one of w_1 and w_2 must be a zero vector. It then follows from

$$\begin{bmatrix} I_n & \Psi \\ \Phi & I_m \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

that w_1 and w_2 are both zero vectors. The contradiction shows that $W = \{0\}$. $\square$

It has been explained in [2] that when K is a regular singular M-matrix with Assumption 1 and $u_1^T v_1 \neq u_2^T v_2$, doubling algorithms (such as SDA in [4] and ADDA in [5]) can be used to find the minimal nonnegative solutions Φ and Ψ simultaneously. But before Theorem 2 is proved in this note, there were a few subtle issues associated with the doubling algorithms. We now have the following modification of [2, Theorem 15] about the ADDA, which uses two parameters α and β. The ADDA is reduced to the SDA when $\alpha = \beta$.

Theorem 3. *Let K be a regular singular M-matrix with Assumption 1 and $u_1^T v_1 \neq u_2^T v_2$. Assume that $\alpha \geq \max a_{ii} > 0$ and $\beta \geq \max d_{ii} > 0$. Then the ADDA is well defined with $I - G_k H_k$ and $I - H_k G_k$ being nonsingular M-matrices for each $k \geq 0$. Moreover, $E_0 \leq 0, F_0 \leq 0$, $E_k \geq 0, F_k \geq 0$, $0 \leq H_{k-1} \leq H_k \leq \Phi$, $0 \leq G_{k-1} \leq G_k \leq \Psi$ for all $k \geq 1$, and*

$$\limsup_{k\to\infty} \sqrt[2^k]{\|H_k - \Phi\|} \leq r(\alpha, \beta), \quad \limsup_{k\to\infty} \sqrt[2^k]{\|G_k - \Psi\|} \leq r(\alpha, \beta),$$

where $r(\alpha, \beta) = \rho\big((R + \alpha I)^{-1}(R - \beta I)\big) \cdot \rho\big((S + \beta I)^{-1}(S - \alpha I)\big) < 1$ with $R = D - C\Phi, S = A - B\Psi$.

Since we have now proved that $I - \Phi\Psi$ and $I - \Psi\Phi$ are nonsingular M-matrices we can use the approach in [3] to prove that the ADDA is well defined with $I - G_k H_k$ and $I - H_k G_k$ being nonsingular M-matrices for each $k \geq 0$ even when $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$. That $r(\alpha, \beta) < 1$ in Theorem 3 is already known in [2]. Thus, H_k converges to Φ quadratically, and G_k converges to Ψ quadratically.

By [5, Theorem 2.3], the parameters $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$ minimize $r(\alpha, \beta)$ among all parameters $\alpha \geq \max a_{ii}$ and $\beta \geq \max d_{ii}$. Therefore, we should normally use the optimal values $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$ for the ADDA. Before Theorem 2 is proved, we avoid using $\alpha = \max a_{ii}$ and $\beta = \max d_{ii}$, to ensure that the ADDA is well defined.

The matrices $(I - G_k H_k)^{-1}$ and $(I - H_k G_k)^{-1}$ appear in the ADDA. With the proof of Theorem 2, we now know that, in Theorem 3, the matrices $I - G_k H_k$ and $I - H_k G_k$ will not converge to singular matrices. This is of course favorable for the ADDA.

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